J. K. SHAH CLASSES/ SYJC MATHS- I SOLUTION: SET 2

Q1. ATTEMPT 6 OUT OF 8

01. $\int \cos^2 x \, dx$

$1 + \cos 2x = 2\cos^2 x$

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$

BACK IN THE SUM

$= \int \dfrac{1 + \cos 2x}{2} \, dx$

$= \dfrac{1}{2}\left(x + \dfrac{\sin 2x}{2}\right) + c$

$= \dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$

02.

$\int \dfrac{1}{x^2 + 8x + 20} \, dx$

$\left(\dfrac{1}{2}(8)\right)^2 = 16$

$= \int \dfrac{1}{x^2 + 8x + 16 + 20 - 16} \, dx$

$= \int \dfrac{1}{(x+4)^2 + 4} \, dx$

$= \int \dfrac{1}{(x+4)^2 + 2^2} \, dx$

$= \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} + c$

$= \dfrac{1}{2} \tan^{-1} \dfrac{x+4}{2} + c$

03.

$\int \dfrac{10x^9 + 10^x.\log 10 \, dx}{x^{10} + 10^x}$

LET

$x^{10} + 10^x = t$

$10x^9 + 10^x.\log 10 \;.\, dx = dt$

NOW THE SUM IS

$= \int \dfrac{1}{t} \, dt$

$= \log |t| + c$

RESUBS.

$= \log | x^{10} + 10^x | + c$

04.

$\int \dfrac{1}{\sqrt{4x^2 - 9}} \, dx$

$= \int \dfrac{1}{\sqrt{(2x)^2 - 3^2}} \, dx$

$= \log \left| x + \sqrt{x^2 - a^2} \right| + c$

$= \dfrac{1}{2} \log \left| 2x + \sqrt{(2x)^2 - 3^2} \right| + c$

COEFF. OF X

$= \dfrac{1}{2} \log \left| 2x + \sqrt{4x^2 - 9} \right| + c$

05.

$\int e^x \left(\dfrac{\cos x + \sin x}{\cos^2 x}\right) dx$

$= \int e^x \left(\dfrac{\cos x}{\cos^2 x} + \dfrac{\sin x}{\cos^2 x}\right) dx$

$= \int e^x \left(\sec x + \sec x.\tan x\right) dx$

$\dfrac{d}{dx} \sec x = \sec x \,.\, \tan x$

HENCE THE SUM IS

$= \int e^x \left(f(x) + f'(x)\right) dx$

$= e^x f(x) + c$

$$= e^x \sec x + c$$

06.

$$\int x \cdot \cos x \, dx$$

$$= x\int \cos x \, dx - \int\left(\frac{d}{dx} x \int \cos x \, dx\right) dx$$

$$= x.\sin x - \int 1 \cdot \sin x \, dx$$

$$= x.\sin x - (-\cos x) + c$$

$$= x.\sin x + \cos x + c$$

07.

$$\int x \cdot \log x \, dx$$

$$= \int \log x \cdot x \, dx$$

$$= \log x \int x \, dx - \int\left(\frac{d}{dx} \log x \int x \, dx\right) dx$$

$$= \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2}\int x \, dx$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4} + c$$

08. $\int \sqrt{9x^2 - 4} \, dx$

$$= \int\sqrt{(3x)^2 - 2^2} \, dx$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left| x + \sqrt{x^2 - a^2} \right| + c$$

$$= \frac{1}{3}\left(\frac{3x}{2}\sqrt{(3x)^2 - 2^2} - \frac{2^2}{2} \log\left| 3x + (3x)^2 - 2^2 \right|\right)$$

COEFF OF X

$$= \frac{x}{2}\sqrt{9x^2 - 4} - \frac{2}{3} \log\left| 3x + \sqrt{9x^2 - 4} \right| + c$$

Q2. A ATTEMPT ANY 2 OUT OF 3

01. $\int \frac{\tan x}{\sec x + \tan x} dx$

$$= \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{\sin x}{1 + \sin x} \times \frac{1 - \sin x}{1 - \sin x} dx$$

$$= \int \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx$$

$$= \int \frac{\sin x - \sin^2 x dx}{\cos^2 x}$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right) dx$$

$$= \int (\sec x \cdot \tan x - \tan^2 x) \quad dx$$

$$= \int \left((\sec x . \tan x - (\sec^2 x - 1)\right) dx$$

$$= \int (\sec x . \tan x - \sec^2 x + 1 \quad dx$$

$$= \sec x - \tan x + x \; c$$

02. $\int e^x \left(\frac{x + 3}{(x + 4)^2}\right) dx$

$$= \int e^x \left(\frac{x + 4 - 1}{(x + 4)^2}\right) dx$$

$$= \int e^x \left(\frac{x + 4}{(x + 4)^2} - \frac{1}{(x + 4)^2}\right) dx$$

$$= \int e^x \left(\frac{1}{x + 4} + \frac{-1}{(x + 4)^2}\right) dx$$

$$\frac{d}{dx} \frac{1}{x + 4} = \frac{-1}{(x + 4)^2}$$

HENCE THE SUM IS

$$= \int e^x \left[f(x) + f'(x)\right] dx$$

$$= e^x \; f(x) + c$$

$$= e^x \frac{1}{x+4} + c$$

$$= \frac{e^x}{x+4} + c$$

03. $\displaystyle\int \frac{x^2}{\sqrt{x^6 + 2x^3 + 3}} dx$

PUT $x^3 = t$

$3x^2 \, . \, dx = dt$

$x^2 \, . \, dx = \dfrac{dt}{3}$

THE SUM IS

$$= \frac{1}{3} \int \frac{1}{\sqrt{t^2 + 2t + 3}} dt$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{t^2 + 2t + 1 + 3 - 1}} dt$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{(t+1)^2 + 2}} dt$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{(t+1)^2 + \sqrt{2}^2}} dt$$

$$= \frac{1}{3} \log \left| t + \sqrt{t^2 + a^2} \right| + c$$

$$= \frac{1}{3} \log \left| t + 1 + \sqrt{(t+1)^2 + \sqrt{2}^2} \right| + c$$

$$= \frac{1}{3} \log \left| t + 1 + \sqrt{t^2 + 2t + 3} \right| + c$$

$$= \frac{1}{3} \log \left| x^3 + 1 + \sqrt{x^6 + 2x^3 + 3} \right| + c$$

Q2. B ATTEMPT ANY 2 OUT OF 3

01. $\displaystyle\int \frac{\sec^2 x}{\sec^2 x - 3 \tan x + 1} dx$

$$= \int \frac{\sec^2 x}{1 + \tan^2 x - 3 \tan x + 1} dx$$

$$= \int \frac{\sec^2 x}{\tan^2 x - 3 \tan x + 2} dx$$

PUT $\tan x = t$

$\sec^2 x \, . \, dx = dt$

THE SUM IS

$$= \int \frac{1}{t^2 - 3t + 2} dt$$

$$\left(\frac{1}{2}(3)\right)^2 = \frac{9}{4}$$

$$= \int \frac{1}{t^2 - 3t + \frac{9}{4} + 2 - \frac{9}{4}} dt$$

$$= \int \frac{1}{\left(t - \frac{3}{2}\right)^2 + \frac{8 - 9}{4}} dt$$

$$= \int \frac{1}{\left(t - \frac{3}{2}\right)^2 - \frac{1}{4}} dt$$

$$= \int \frac{1}{\left(t - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dt$$

$$= \frac{1}{2a} \log \left| \frac{t - a}{t + a} \right| + c$$

$$= \frac{1}{2\,\frac{1}{2}} \log \left| \frac{t - \frac{3}{2} - \frac{1}{2}}{t - \frac{3}{2} + \frac{1}{2}} \right| + c$$

$$= \log \left| \frac{2t - 3 - 1}{2t - 3 + 1} \right| + c$$

$$= \log \left| \frac{2t - 4}{2t - 2} \right| + c$$

$$= \log \left| \frac{t - 2}{t - 1} \right| + c$$

$$= \log \left| \frac{\tan x - 2}{\tan x - 1} \right| + c$$

02. $\int x.\tan^{-1}x\ dx$

$$= \int \tan^{-1}x.\ x\ dx$$

$$= \tan^{-1}x \int x\ dx - \int \left(\frac{d}{dx}\tan^{-1}x \int x\ dx \right) dx$$

$$= \tan^{-1}x\ .\ \frac{x^2}{2} - \int \frac{1}{1+x^2}\ .\ \frac{x^2}{2}\ dx$$

$$= \frac{x^2}{2}.\ \tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2.}\ dx$$

$$= \frac{x^2}{2}.\ \tan^{-1}x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2.}\ .\ dx$$

$$= \frac{x^2}{2}.\ \tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2.}\right) dx$$

$$= \frac{x^2}{2}.\ \tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + c$$

$$= \frac{x^2}{2}.\ \tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + c$$

03. $\int \frac{x^2+2}{(x^2+4)(x^2+9)}\ dx$

SOLUTION

$$\frac{x^2+2}{(x^2+4)(x^2+9)} = \frac{A}{x^2+4} + \frac{B}{x^2+9}$$

x^2 = t (say)

$$\frac{t+2}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9}$$

$$t+2 = A(t+9) + B(t+4)$$

Put $t = -9$

$$-9+2 = B(-9+4)$$

$$-7 = B(-5) \qquad \therefore\ B = \frac{7}{5}$$

Put $t = -4$

$$-4+2 = A(-4+9)$$

$$-2 = A(5) \qquad \therefore\ A = \frac{-2}{5}$$

THEREFORE

$$\frac{t+2}{(t+4)(t+9)} = \frac{\frac{-2}{5}}{t+4} + \frac{\frac{7}{5}}{t+9}$$

HENCE

$$\frac{x^2+2}{(x^2+4)(x^2+9)} = \frac{\frac{-2}{5}}{x^2+4} + \frac{\frac{7}{5}}{x^2+9}$$

BACK IN THE SUM

$$= \int \frac{\frac{-2}{5}}{x^2+4} + \frac{\frac{7}{5}}{x^2+9}\, dx$$

$$= \int \frac{\frac{-2}{5}}{x^2+2^2} + \frac{\frac{7}{5}}{x^2+3^2}\, dx$$

$$= \frac{-2}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{7}{5}\ \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + c$$

$$= \frac{-1}{5} \tan^{-1}\left(\frac{x}{2}\right) + \frac{7}{15} \tan^{-1}\left(\frac{x}{3}\right) + c$$

Q3. A ATTEMPT ANY 2 OUT OF 3

01.

$$I = \int_0^3 \frac{\sqrt[3]{x+4}}{\sqrt[3]{x+4} + \sqrt[3]{7-x}}\, dx \quad \text{.....(1)}$$

USING $\int_0^a f(x)dx = \int_0^a f(a-x)\, dx$

CHANGE 'x' TO '3 – x'

$$I = \int_0^3 \frac{\sqrt[3]{3-x+4}}{\sqrt[3]{3-x+4} + \sqrt[3]{7-(3-x)}}\, dx$$

$$I = \int_0^3 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{7-3+x}}\, dx$$

$$I = \int_0^3 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{4+x}}\, dx$$

$$I = \int_0^3 \frac{\sqrt[3]{7-x}}{\sqrt[3]{7-x} + \sqrt[3]{x+4}}\, dx \quad \text{...... (2)}$$

(1) + (2)

$$2I = \int_0^3 \frac{\sqrt[3]{x+4} + \sqrt[3]{7-x}}{\sqrt[3]{x+4} + \sqrt[3]{7-x}}\, dx$$

$$2I = \int_0^3 1\ dx$$

$$2I = \left(x\right)_0^3$$

$$2I = 3 - 0$$

$$I = 3/2$$

02.

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt[3]{\cot x}}\, dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}}\, dx$$

USING $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$

WE CHANGE 'x' TO '$\pi/6 + \pi/3$ **− x' i.e.**

WE CHANGE 'x' TO '$\pi/2 - x$'

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin(\pi/2-x)}}{\sqrt[3]{\sin(\pi/2-x)} + \sqrt[3]{\cos(\pi/2-x)}}\, dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}}\, dx \quad \textbf{... (2)}$$

(1) + (2)

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}}\, dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1\, dx$$

$$2I = \Big(x \Big)_{\pi/6}^{\pi/3}$$

$$2I = \frac{\pi}{3} - \frac{\pi}{6}$$

$$2I = \frac{2\pi - \pi}{6}$$

$$I = \frac{\pi}{12}$$

03.

$$\int_0^a \frac{1}{\sqrt{ax - x^2}}\, dx$$

$$= \int_0^a \frac{1}{\sqrt{0 - (x^2 - ax + a^2/4) + a^2/4}}\, dx$$

$$= \int_0^a \frac{1}{\sqrt{a^2/4 - (x - a/2)^2}}\, dx$$

$$= \int_0^a \frac{1}{\sqrt{(a/2)^2 - (x - a/2)^2}}\, dx$$

$$= \left\{ \sin^{-1}\left(\frac{x - a/2}{a/2} \right) \right\}_0^a$$

$$= \left\{ \sin^{-1}\left(\frac{2x - a}{a} \right) \right\}_0^a$$

$$= \sin^{-1}\left(\frac{2a - a}{a} \right) - \sin^{-1}\left(\frac{0 - a}{a} \right)$$

$$= \sin^{-1}\left(\frac{a}{a} \right) - \sin^{-1}\left(\frac{-a}{a} \right)$$

$$= \sin^{-1} 1 - \sin^{-1}(-1)$$

$$= \sin^{-1} 1 + \sin^{-1} 1$$

$$= 2 \sin^{-1} 1$$

$$= 2. \frac{\pi}{2}$$

$$= \pi$$

Q3. B ATTEMPT ANY 2 OUT OF 3

01.

$$I = \int_0^4 \frac{1}{x + \sqrt{16 - x^2}}\, dx$$

Put $x = 4\sin\theta$ $\frac{dx}{d\theta} = 4\cos\theta$ $dx = 4\cos\theta\, d\theta$

when $x = 4$, $\sin\theta = 1$, $\theta = \pi/2$

when $x = 0$, $\sin\theta = 0$, $\theta = 0$

$$I = \int_0^{\pi/2} \frac{1}{4\sin\theta + \sqrt{16 - 16\sin^2\theta}}\, 4\cos\theta\, d\theta$$

$$I = \int_0^{\pi/2} \frac{4\cos\theta}{4\sin\theta + \sqrt{16(1 - \sin^2\theta)}}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{4\cos\theta}{4\sin\theta + \sqrt{16\cos^2\theta}}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{4\cos\theta}{4\sin\theta + 4\cos\theta}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta}\, d\theta \quad \textbf{...... (1)}$$

USING $\int_0^a f(x)dx = \int_0^a f(a - x)\, dx$

WE CHANGE 'θ' **TO** '$\pi/2 - \theta$'

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - \theta)}{\cos(\pi/2 - \theta) + \sin(\pi/2 - \theta)}\, d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta}\, d\theta \quad \textbf{...... (2)}$$

(1) + (2)

$$I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}\, d\theta$$

$$2I = \int_0^{\pi/2} 1\, d\theta$$

$$2I = \left(\theta\right)_0^{\pi/2}$$

$$2I = \pi/2 - 0$$

$$2I = \pi/2$$

$$I = \pi/4$$

02.

$$I = \int_0^2 x^2(2 - x)^{1/2}\, dx$$

USING $\int_0^a f(x)dx = \int_0^a f(a - x)\, dx$

WE CHANGE 'x' **TO** '$2 - x$'

$$I = \int_0^2 (2 - x)^2 \,.\, x^{1/2}\, dx$$

$$I = \int_0^2 (4 - 4x + x^2).\, x^{1/2}\, dx$$

$$I = \int_0^2 (4x^{1/2} - 4x^{3/2} + x^{5/2})\, dx$$

$$I = \left(4\frac{x^{3/2}}{\frac{3}{2}} - 4\frac{x^{5/2}}{\frac{5}{2}} + \frac{x^{7/2}}{\frac{7}{2}} \right)_0^2$$

$$I = \left(\frac{8}{3}x^{3/2} - \frac{8}{5}x^{5/2} + \frac{2x^{7/2}}{7} \right)_0^2$$

$$I = \frac{8}{3} 2^{3/2} - \frac{8}{5} 2^{5/2} + \frac{2}{7} 2^{7/2}$$

$$I = \frac{8}{3} 2\sqrt{2} - \frac{8}{5} 2^{2}\sqrt{2} + \frac{2}{7} 2^{3}\sqrt{2}$$

$$I = \frac{16\sqrt{2}}{3} - \frac{32\sqrt{2}}{5} + \frac{16\sqrt{2}}{7}$$

$$I = 16\sqrt{2}\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right)$$

$$I = 16\sqrt{2} \; \frac{35 - 42 + 15}{105}$$

$$I = 16\sqrt{2} \; \frac{8}{105}$$

$$I = \frac{128\sqrt{2}}{105}$$

03. $$\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(2 + \sin x)} dx$$

$\sin x = t$ $\quad x = 0 \; , t = \sin 0 = 0$

$\cos x \, . \, dx = dt$ $\quad x = \frac{\pi}{2}, \; t = \sin \frac{\pi}{2} = 1$

THE SUM IS

$$\int_0^1 \frac{1}{(1 + t)(2 + t)} dt$$

$$\frac{1}{(1 + t)(2 + t)} = \frac{A}{1 + t} + \frac{B}{2 + t}$$

$$1 = A(2 + t) + B(1 + t)$$

Put $t = -2$

$$1 = B(1 - 2)$$

$$1 = B(-1)$$

$$-1 = B$$

Put $t = -1$

$$1 = A(2 - 1)$$

$$1 = A(1)$$

$$1 = A$$

HENCE

$$\frac{1}{(1 + t)(2 + t)} = \frac{1}{1 + t} + \frac{-1}{2 + t}$$

BACK IN THE SUM

$$= \int_0^1 \frac{1}{1 + t} - \frac{1}{2 + t} \; dt$$

$$= \Big(\log |1 + t| - \log |2 + t| \Big)_0^1$$

$$= \left(\log \left| \frac{1 + t}{2 + t} \right| \right)_0^1$$

$$= \left(\log \frac{2}{3}\right) - \left(\log \frac{1}{2}\right)$$

$$= \log \left(\frac{\frac{2}{3}}{\frac{1}{2}} \right) = \log \frac{4}{3}$$