J.K. SHAH CLASSES

MATHEMATICS & STATISTICS

SYJC TEST – 03 – SET 2

DURATION – 1 $^{1}/_{2}$ HR **MARKS – 40**

TOPIC : DIFFERENTIATION + RATIO – PROPORTION + PARTNERSHIP + COMMERCIAL ARITHMETIC (PART)

SECTION - I

Q1. Attempt any THREE of the following (2 marks each) **(6 marks)**

01. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$ Find dy/dx

SOLUTION SET

Solution

Put $x = \sin\theta$

$y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$y = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$y = \sin^{-1}\left(2\sin\theta \cdot \cos\theta\right)$

$y = \sin^{-1} \sin 2\theta$

$y = 2\theta$

$y = 2\sin^{-1} x$

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

02. $y = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$ Find dy/dx

Solution

$y = \tan^{-1}\left(\frac{3x + 2x}{1 - 3x.2x}\right)$

$y = \tan^{-1} 3x + \tan^{-1} 2x$

$$\frac{dy}{dx} = \frac{1}{1+9x^2}\cdot\frac{d}{dx}(3x) + \frac{1}{1+4x^2}\cdot\frac{d}{dx}(2x)$$

$$\frac{dy}{dx} = \frac{3}{1 + 9x^2} + \frac{2}{1 + 4x^2}$$

03. $y = \tan^{-1}\left(\frac{1 - \cos x}{\sin x}\right)$ Find dy/dx

Solution

$$y = \tan^{-1}\left(\frac{2\sin^2 x/2}{2\sin x/2.\cos x/2}\right)$$

$$y = \tan^{-1}\left(\frac{\sin x/2}{\cos x/2}\right)$$

$$y = \tan^{-1}(\tan x/2)$$

$$y = \frac{x}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}$$

04. $y = x^{\tan^{-1} x}$ Find dy/dx

Solution

Taking log on both sides

$$\text{Log } y = \tan^{-1}x \,.\, \log x$$

Differentiating wrt x

$$\frac{1}{y}\frac{dy}{dx} = \tan^{-1}x.\ \frac{d}{dx}\log x + \log x\ \frac{d}{dx}\tan^{-1}x$$

$$\frac{1}{y}\frac{dy}{dx} = \tan^{-1}x\,.\,\frac{1}{x} + \log x\ \frac{.\ 1}{1 + x^2}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\text{tain}^{-1}x}{x} + \frac{\log x}{1 + x^2}$$

$$\frac{dy}{dx} = y\left(\frac{\tan^{-1}x}{x} + \frac{\log x}{1 + x^2}\right)$$

$$\frac{dy}{dx} = x^{\tan^{-1} x}\left(\frac{\tan^{-1}x}{x} + \frac{\log x}{1 + x^2}\right)$$

Q2. **Attempt any TWO of the following** **(3 marks each)** **(6 marks)**

01. $x^2y^k = (x + y)^{2+k}$. Show that : $\frac{dy}{dx} = \frac{y}{x}$

Solution :
taking log on both sides

$2 \log x + k \log y = (2 + k) \log (x + y)$

Differentiating wrt x

$$2 \frac{1}{x} + k \frac{1}{y} \frac{dy}{dx} = 2+k \frac{1}{x + y} \frac{d}{dx}(x + y)$$

$$\frac{2}{x} + \frac{k}{y} \frac{dy}{dx} = \frac{2 + k}{x + y} \left(1 + \frac{dy}{dx}\right)$$

$$\frac{2}{x} + \frac{k}{y} \frac{dy}{dx} = \frac{2 + k}{x + y} + \frac{2 + k}{x + y} \frac{dy}{dx}$$

$$\left(\frac{k}{y} - \frac{2 + k}{x + y}\right) \frac{dy}{dx} = \frac{2 + k}{x + y} - \frac{2}{x}$$

CONT.

$$\frac{kx + ky - 2y - ky}{y.(x + y)} \frac{dy}{dx} = \frac{2x + kx - 2x - 2y}{x.(x + y)}$$

$$\frac{kx - 2y}{y(x + y)} \frac{dy}{dx} = \frac{kx - 2y}{x(x + y)}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

02. $y = (\tan x)^x$ Find dy/dx

Solution

Taking log on both sides

$\text{Log } y = x \cdot \log (\tan x)$

Differentiating wrt x

$$\frac{1}{y} \frac{dy}{dx} = x \frac{d}{dx} \log (\tan x) + \log (\tan x) \frac{d}{dx} x$$

$$\frac{1}{y} \frac{dy}{dx} = x \frac{1}{\tan x} \frac{d}{dx} \tan x + \log (\tan x) \cdot 1$$

$$\frac{1}{y} \frac{dy}{dx} = x \frac{1}{\tan x} \cdot \sec^2 x + \log (\tan x)$$

cont.

$$\frac{1}{y} \frac{dy}{dx} = x. \frac{1}{\frac{\sin x}{\cos x}} \frac{1}{\cos^2 x} + \log (\tan x)$$

$$\frac{1}{y} \frac{dy}{dx} = x. \frac{1}{\sin x \cdot \cos x} + \log (\tan x)$$

$$\frac{1}{y} \frac{dy}{dx} = x. \frac{2}{2 \sin x \cdot \cos x} + \log (\tan x)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{2x}{\sin 2x} + \log (\tan x)$$

$$\frac{dy}{dx} = y \left(\frac{2x}{\sin 2x} + \log (\tan x)\right)$$

$$\frac{dy}{dx} = (\tan x)^x \left(\frac{2x}{\sin 2x} + \log (\tan x)\right)$$

03. $x^y = e^x$ Show that : $\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$

Solution

$x^y = e^x$

Taking log on both sides

$y \cdot \log x = x \cdot \log e$

$y \cdot \log x = x$

$y = \frac{x}{\log x}$

Differentiating wrt x

$\frac{dy}{dx} = \frac{\log x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx} \log x}{(\log x)^2}$

$\frac{dy}{dx} = \frac{\log x \cdot 1 - x \cdot \frac{1}{x}}{(\log x)^2}$

$\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$ …….. proved

cont.

Q3. Attempt any TWO of the following (4 marks each) **(8 marks)**

01. $y = x^x + x^{\sin x}$ Find dy/dx

Solution

$y = u + v$

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dv}$ ……..(1)

Now

$u = x^x$

Taking log on both sides

$\text{Log } u = x \cdot \log x$

DIFF WRT X

$\frac{1}{u}\frac{du}{dx} = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$

$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$

$\frac{1}{u}\frac{du}{dx} = 1 + \log x$

$\frac{du}{dx} = u (1 + \log x)$

$\frac{du}{dx} = x^x (1 + \log x)$ ….. (2)

$v = x^{\sin x}$

Taking log on both sides

$\text{Log } y = \sin x \cdot \log x$

DIFF WRT X

$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{d}{dx} \log x + \log x \cdot \frac{d}{dx} \sin x$

$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x$

$\frac{1}{v}\frac{dv}{dx} = \frac{\sin x}{x} + \cos x \cdot \log x$

$\frac{dv}{dx} = v \frac{\sin x}{x} + \cos x \cdot \log x$

$\frac{dv}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \cos x \cdot \log x \right)$ …(3)

cont.

Subs (2) & (3) in (1)

$\therefore \frac{dy}{dx} = x^x (1 + \log x) + x^{\sin x} \left(\frac{\sin x}{x} + \cos x \cdot \log x \right)$

02. $\log \left(\frac{x^4 - y^4}{x^4 + y^4}\right) = k$ Show that : $\frac{dy}{dx} = \frac{y}{x}$

Solution

$$\log \frac{x^4 - y^4}{x^4 + y^4} = k$$

$$\frac{x^4 - y^4}{x^4 + y^4} = e^k$$

$$\frac{x^4 - y^4}{x^4 + y^4} = m$$

$$x^4 - y^4 = m(x^4 + y^4)$$

$$x^4 - y^4 = mx^4 + my^4$$

$$x^4 - mx^4 = y^4 + my^4$$

$$x^4 (1 - m) = y^4 (1 + m)$$

$$y^4 = x^4 \frac{1 - m}{1 + m} \quad \ldots\ldots. \quad (1)$$

Differentiating wrt x

$$4y^3 \frac{dy}{dx} = 4x^3 \frac{1 - m}{1 + m}$$

$$\frac{dy}{dx} = \frac{x^3}{y^3} \frac{1 - m}{1 + m}$$

$$\frac{dy}{dx} = \frac{x^3}{y^3} \frac{y^4}{x^4} \quad \ldots\ldots \text{ from (1)}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

03. $y = \tan^{-1} (\sec x + \tan x)$. Find dy/dx

Solution

$$y = \tan^{-1} \left[\frac{1 + \sin x}{\cos x}\right]$$

$$y = \tan^{-1} \left[\frac{\cos^2 x/2 + \sin^2 x/2 + 2\sin x/2 . \cos x/2}{\cos^2 x/2 - \sin^2 x/2}\right]$$

$$y = \tan^{-1} \left[\frac{(\cos x/2 + \sin x/2)^2}{(\cos x/2 - \sin x/2)(\cos x/2 + \sin x/2)}\right]$$

$$y = \tan^{-1}\left[\frac{\cos {}^{x}/_{2} + \sin {}^{x}/_{2}}{\cos {}^{x}/_{2} - \sin {}^{x}/_{2}}\right]$$

$$y = \tan^{-1}\left(\frac{\dfrac{\cos {}^{x}/_{2} + \sin {}^{x}/_{2}}{\cos {}^{x}/_{2}}}{\dfrac{\cos {}^{x}/_{2} - \sin {}^{x}/_{2}}{\cos {}^{x}/_{2}}}\right)$$

$$y = \tan^{-1}\left[\frac{1 + \tan {}^{x}/_{2}}{1 - \tan {}^{x}/_{2}}\right]$$

$$y = \tan^{-1} 1 + \tan^{-1}(\tan {}^{x}/_{2})$$

$$y = \tan^{-1} 1 + {}^{x}/_{2}$$

$$\frac{dy}{dx} = 0 + \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}$$

SECTION - II

Q4. Attempt any THREE of the following (2 marks each) **(6 marks)**

01. Raghu , Madhu and Ramu started a business in partnership by investing ₹ 60,000 , ₹ 40,000 and ₹ 75,000 respectively . At the end of the year they found that they have incurred a loss of ₹ 24,500 . Find how much loss each one had to bear

SOLUTION

STEP 1 :

Loss will be shared in the

'RATIO OF THE INVESTMENT'

	RAHGU		MADHU		RAMU	
=	60,000	:	40,000	:	75,000	
=	60	:	40	:	75	
=	12	:	8	:	15	TOTAL = 35

STEP 2 :

LOSS = ₹ 24,500

Raghu's share of loss $= \frac{12}{35} \times \frac{700}{\cancel{24,500}} = ₹\ 8,400$

Madhu's share of loss $= \frac{8}{35} \times \frac{700}{\cancel{24,500}} = ₹\ 5,600$

Ramu's share of loss $= \frac{15}{35} \times \frac{700}{\cancel{24,500}} = ₹\ 10,500$

02. the wholesaler allows 25% trade discount and 5% cash discount . What will be the net price of an article marked at ₹ 1600

SOLUTION

List Price	= ₹	1,600
Less 25% T.D.: $\frac{25}{100} \times 1600$	–	400
Invoice Price	= ₹	1,200
Less 5% C.D.: $\frac{5}{100} \times 1200$	–	60
Net Selling Price	= ₹	1,140

03. a salesman gets a commission of 6.5% on the total value of sales made by him and additional bonus of 0.25% on the excess of his sales over ₹ 16,000 . Find his total income on a turnover of ₹ 25,000

SOLUTION

Sale = ₹ 25,000

a salesman gets a commission of 6.5% on the total value of sales made by him and additional bonus of 0.25% on the excess of his sales over □ 16,000

∴ His total income

$$= \frac{6.5}{100}(25{,}000) + \frac{0.25}{100}(25{,}000 - 16{,}000)$$

$$= \frac{6.5}{100}(25{,}000) + \frac{25}{10000}(9{,}000)$$

$$= 1{,}625 + 22.50$$

$$= ₹\ 1{,}647.50$$

04. the income of an agent remains unchanged though the rate of commission is increased from 5% to 6.25% . Find the percentage reduction in the value of business

SOLUTION

Let initial sales = ₹ 100

Rate of commission = 5%

∴ Commission = ₹ 5

Let the new sales = ₹ x

Rate of commission = 6.25%

∴ Commission $= \frac{6.25x}{100}$

Since the income of the broker remains unchanged

$$\frac{6.25x}{100} = 5$$

$$x = \frac{5 \times 10000}{625}$$

$$x = 80$$

∴ new sales = ₹ 80

Hence the percentage reduction in the value of the business is 20%

Q5. Attempt any TWO of the following (3 marks each) **(6 marks)**

01. After allowing 20% trade discount and 5% cash discount , a television set was sold for ₹ 9,120 . What was the catalogue price of the set

SOLUTION

List Price	= ₹	100
Less 20% T.D.	–	20
Invoice Price	= ₹	80
Less 5% C.D.	–	4
Net Selling Price	= ₹	76

Now When ;

Net SP = 76 ; Cat. Price = 100

Net SP = 9120 ; Cat. Price $= \frac{120}{76} \times 100$

= ₹ 12,000

02. The ratio of number of boys and girls in a school is 3 : 2 . If 20 % of the boys and 30% of the girls are scholarship holders , find the percentage of students who are not scholarship holders

SOLUTION

Let boys = 3x & girls = 2x , Total = 5x

Scholarship holders $= \frac{20}{100}(3x) + \frac{30}{100}(2x)$

$= \frac{3x}{5} + \frac{3x}{5}$

$= \frac{6x}{5}$

Non scholarship holders $= 5x - \frac{6x}{5} = \frac{19x}{5}$

Hence percentage number of students who are not scholarship holders

$= \frac{\frac{19x}{5}}{5x} \times 100$

$= \frac{19}{25} \times 100$

= 76%

03. the ratio of prices of two houses are 2:3 . Two years later when price of first house has increased by 30% and that of second house by ₹ 90,000 the ratio of prices becomes 5:7 . Find original prices of two houses

SOLUTION

Let price of 1st house = 2x

Price of 2nd house = 3x

As per the given condition

$$\frac{2x + \frac{30}{100}(2x)}{3x + 90000} = \frac{5}{7}$$

$$\frac{13x}{5} = \frac{5}{7}(3x + 90000)$$

$$91x = 25(3x + 90000)$$

$$91x = 75x + 2250000$$

$$16x = 2250000$$

$$x = 140625$$

price of 1st house = 2(140625) = ₹ 2,81,250

Price of 2nd house = 3(140625) = ₹ 4,21,875

Q6. Attempt any TWO of the following (4 marks each) **(8 marks)**

01. Three persons A , B , C whose salaries amount to ₹ 21000 . Their savings are 20% , 30% and 40% of their salaries respectively . If their expenditures are in the 8 : 14 : 3 , find their respective salaries

SOLUTION

STEP 1

	A	B	C
Salary	x	y	z
Savings	$\frac{20x}{100}$	$\frac{30y}{100}$	$\frac{40z}{100}$
Expenses	$\frac{80x}{100}$	$\frac{70y}{100}$	$\frac{60z}{100}$

STEP 2

their expenditures are in the 8 : 14 : 3

$\frac{\text{A's expense}}{\text{B's expense}} = \frac{8}{14}$ | $\frac{\text{B's expense}}{\text{C's expense}} = \frac{14}{3}$

$\frac{\frac{80x}{100}}{\frac{70y}{100}} = \frac{8}{14}$ | $\frac{\frac{70y}{100}}{\frac{60z}{100}} = \frac{14}{3}$

$\frac{8x}{7y} = \frac{8}{14}$ | $\frac{7y}{6z} = \frac{14}{3}$

$\frac{x}{y} = \frac{1}{2}$ | $\frac{y}{z} = \frac{4}{1}$

STEP 3 : RATIO OF SALARIES

x	:	y	:	z
1^{x2}	:	2^{x2}		
		4	:	1
2	:	4	:	1

STEP 4 SALARIES

TOTAL OF SALARIES = 21,000

A's salary = $\frac{2}{7}$ x 21000 = ₹ 6,000

B's salary = $\frac{4}{7}$ x 21000 = ₹ 12,000

C's Salary = $\frac{1}{7}$ x 21000 = ₹ 3,000

02. P and Q started a business with capitals in the ratio 4 : 3 . After 9 months P withdrew 25% of his capital and Q put in an equal amount in addition to his earlier capital . If at the end of the year P's share in the profit was ₹ 15,450 , find the total profit and Q's share of profit

SOLUTION

PARTNER's NAME	CAPITAL INVESTED	PERIOD OF INVESTMENT
P	₹ 4k	9 MONTHS
	– 25%	
	₹ 3k	3 MONTHS
Q	₹ 3k + k	9 MONTHS
	₹ 4k	3 MONTHS

STEP 1 :

Profits will be shared in the
'RATIO OF PRODUCT OF CAPITAL INVESTED & PERIOD OF INVESTMENT'

	P		Q	
=	4k x 9 + 3k x 3	:	3k x 9 + 4k x 3	
=	36k + 9k	:	27k + 12k	
=	45k	:	39k	
=	15	:	13	TOTAL = 28

STEP 2 :

P share of profit = ₹ 15,450

P's share of profit = $\frac{15}{28}$ x Total Profit

15,450 = $\frac{15}{28}$ x Total Profit

Total Profit = $\frac{15450 \times 28^{1030}}{15}$

= ₹ 28,840

Q's share of profit = $\frac{13}{28}$ x $\frac{28,840^{1030}}{}$

= ₹ 13,390

03. X and Y are partners in a business with their capitals as ₹ 2,00,000 and ₹ 3,00,000 respectively . Z wishes to join the business with a capital of ₹ 2,00,000 at the beginning of the financial year . They agree that the goodwill be taken as twice the average annual for the last three years . Last three years profits are ₹ 25,000 , ₹ 40,000 and ₹ 40,000 respectively . Find the goodwill amount that Z would be required to pay X and Y separately

SOLUTION :

STEP 1 :

Ratio of Investment

X		Y		Z
2,00,000	:	3,00,000	:	2,00,000

= 2 : 3 : 2

STEP 2 :

Good will amount = $2 \frac{25,000 + 40,000 + 40,000}{3}$

= $2 \frac{1,05,000}{3}$

= 2 (35,000)

= ₹ 70,000

STEP 3 :

Z's share in good will amount

= $\frac{2}{7}$ x 70,000

= ₹ 20,000

STEP 4 :

Z has to pay above good will amount of ₹ 20,000 to X and Y in the ratio of their investment

Z pays to X = $\frac{2}{5}$ x 20,000 = ₹ 8,000

Z pays to Y = $\frac{3}{5}$ x 20,000 = ₹ 12,000