EXERCISE - 6.4

$$\int \frac{px + q}{ax^2 + bx + c}\, dx$$

REMEMBER THIS EXERCISE AS

LINEAR / QUADRATIC

Express **Numerator = A $\frac{d}{dx}$(denominator) + B**

$$= \int \frac{A \frac{d}{dx}(\text{denominator}) + B}{ax^2 + bx + c}\, dx$$

$$= A \int \frac{\frac{d}{dx}(\text{denominator})\, dx}{ax^2 + bx + c} + B \int \frac{1}{ax^2 + bx + c}\, dx$$

$$= I_1 + I_2$$

Where

$$I_1 = A \int \frac{\frac{d}{dx}(\text{denominator})\, dx}{ax^2 + bx + c}$$

Using $\int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c$

$$= A \,.\, \log|ax^2 + bx + c| + c_1$$

$$I_2 = B \int \frac{1}{ax^2 + bx + c}\, dx$$

REFER : EX 3A

✓ $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

✓ $\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

01.

I = $\displaystyle\int \frac{x+2}{x^2+2x+2}\,dx$

WE NEED TO EXPRESS

NUMERATOR = A $\frac{d}{dx}$ DENOMIANTOR + B

$x + 2 = A\frac{d}{dx}(x^2+2x+2) + B$

$x + 2 = A(2x+2) + B$

$x + 2 = 2Ax + 2A + B$

On comparing ;

$2A = 1$	$2A + B = 2$
$A = \frac{1}{2}$	$2.\frac{1}{2} + B = 2$
	$1 + B = 2$
	$B = 1$

HENCE

$x + 2 = \frac{1}{2}(2x+2) + 1$

BACK IN THE SUM

$$I = \int \frac{\frac{1}{2}(2x+2) + 1}{x^2+2x+2}\,dx$$

$$I = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx + \int \frac{1}{x^2+2x+2}\,dx$$

$I = I_1 + I_2$

Now

$$I_1 = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx$$

$$I_1 = \frac{1}{2}\int \frac{f'(x)}{f(x)}\,dx$$

$$I_1 = \frac{1}{2}\log|f(x)| + c_1$$

$$I_1 = \frac{1}{2}\log|x^2+2x+2| + c_1$$

Now

$$I_2 = \int \frac{1}{x^2+2x+2}\,dx$$

$$\left(\frac{1}{2}(2)\right)^2 = 1$$

$$= \int \frac{1}{x^2+2x+1+2-1}\,dx$$

$$= \int \frac{1}{(x+1)^2+1}\,dx$$

$$= \int \frac{1}{(x+1)^2+1^2}\,dx$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$= \frac{1}{1}\tan^{-1}\frac{x+1}{1} + c_2$$

$$= \tan^{-1}(x+1) + c_2$$

FINALLY

$$I = \frac{1}{2}\log|x^2+2x+2| + c_1 + \tan^{-1}(x+1) + c_2$$

$$I = \frac{1}{2}\log|x^2+2x+2| + \tan^{-1}(x+1) + c$$

where $c = c_1 + c_2$

02.

I = $\int \frac{x}{x^2 + x + 1} dx$

WE NEED TO EXPRESS

NUMERATOR = $A \frac{d}{dx}$ DENOMIANTOR + B

$x = A \frac{d}{dx}(x^2 + x + 1) + B$

$x = A(2x + 1) + B$

$x = 2Ax + A + B$

On comparing ;

$2A = 1$	$A + B = 0$
$A = \frac{1}{2}$	$\frac{1}{2} + B = 0$
	$B = -\frac{1}{2}$

HENCE

$x = \frac{1}{2}(2x + 1) - \frac{1}{2}$

BACK IN THE SUM

$I = \int \frac{\frac{1}{2}(2x + 1) - \frac{1}{2}}{x^2 + x + 1} dx$

$I = \frac{1}{2}\int \frac{2x + 1}{x^2 + x + 1} dx - \frac{1}{2}\int \frac{1}{x^2 + x + 1} dx$

$I = I_1 - I_2$

Now

$I_1 = \frac{1}{2}\int \frac{2x + 1}{x^2 + x + 1} dx$

$I_1 = \frac{1}{2}\int \frac{f'(x)}{f(x)} dx$

$I_1 = \frac{1}{2} \log | f(x) | + c^1$

$I_1 = \frac{1}{2} \log | x^2 + x + 1 | + c^1$

Now

$I_2 = \frac{1}{2}\int \frac{1}{x^2 + x + 1} dx$

$\left(\frac{1}{2}(1)\right)^2 = \frac{1}{4}$

$= \frac{1}{2}\int \frac{1}{x^2 + x + \frac{1}{4} + 1 - \frac{1}{4}} dx$

$= \frac{1}{2}\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{4 - 1}{4}} dx$

$= \frac{1}{2}\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} dx$

$= \frac{1}{2}\int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$

$= \frac{1}{2} \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

$= \frac{1}{2} \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c^2$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + c^2$

FINALLY

$I = I_1 - I_2$

$I = \frac{1}{2} \log |x^2 + x + 1| + c_1 - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + c_2$

$I = \frac{1}{2} \log |x^2 + x + 1| - \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + c$

where $c = c_1 - c_2$

03.

I = $\displaystyle\int \frac{x - 1}{3x^2 - 4x + 3}\,dx$

WE NEED TO EXPRESS

NUMERATOR = $A \frac{d}{dx}$ DENOMIANTOR + B

$x - 1 = A \frac{d}{dx}(3x^2 - 4x + 3) + B$

$x - 1 = A(6x - 4) + B$

$x - 1 = 6Ax - 4A + B$

On comparing ;

$6A = 1$	$-4A + B = -1$
$A = \frac{1}{6}$	$-4.\frac{1}{6} + B = -1$
	$-\frac{2}{3} + B = -1$
	$B = -1/3$

HENCE

$x - 1 = \frac{1}{6}(6x - 4) - \frac{1}{3}$

BACK IN THE SUM

$$I = \int \frac{\frac{1}{6}(6x - 4) - \frac{1}{3}}{3x^2 - 4x + 3}\,dx$$

$$I = \frac{1}{6}\int \frac{6x - 4}{3x^2-4x+3}\,dx - \frac{1}{3}\int \frac{1}{3x^2-4x+3}\,dx$$

$I = I_1 - I_2$

Now

$$I_1 = \frac{1}{6}\int \frac{6x - 4}{3x^2-4x+3}\,dx$$

$$I_1 = \frac{1}{6}\int \frac{f'(x)}{f(x)}\,dx$$

$$I_1 = \frac{1}{6}\log|f(x)| + c_1$$

$$I_1 = \frac{1}{6}\log|3x^2-4x+3| + c_1$$

Now

$$I_2 = \frac{1}{3}\int \frac{1}{3x^2 - 4x + 3}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\frac{3x^2- 4x + 3}{3}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{x^2 - \frac{4x}{3} + 1}\,dx$$

$$\left(\frac{1}{2}\ \frac{4}{3}\right)^2 = \frac{4}{9}$$

$$= \frac{1}{9}\int \frac{1}{x^2 - \frac{4x}{3} + \frac{4}{9} + 1 - \frac{4}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \frac{9-4}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \frac{5}{9}}\,dx$$

$$= \frac{1}{9}\int \frac{1}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2}\,dx$$

$$= \frac{1}{9}\,\frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$= \frac{1}{9}\,\frac{1}{\frac{\sqrt{5}}{3}}\tan^{-1}\left(\frac{x - \frac{2}{3}}{\frac{\sqrt{5}}{3}}\right) + c_2$$

$$= \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x - 2}{\sqrt{5}}\right) + c_2$$

FINALLY

$I = I_1 - I_2$

$$I = \frac{1}{6}\log|3x^2-4x+3| + c_1 - \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x - 2}{\sqrt{5}}\right) + c_2$$

$$I = \frac{1}{6}\log|3x^2-4x+3| - \frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{3x - 2}{\sqrt{5}}\right) + c$$

where $c = c_1 - c_2$

04. $\int \frac{x^3 + x + 1}{x^2 - 1} dx$

NOTE : Numerator > Denominator .
Divide and rewrite as
Quotient + $\frac{\text{Remainder}}{\text{Divisor}}$

$$\begin{array}{r|l} & x \\ x^2 - 1 & x^3 + x + 1 \\ & x^3 - x \\ & - \quad + \\ & \quad 2x + 1 \end{array}$$

Back into the sum

$$\int x + \frac{2x + 1}{x^2 - 1} dx$$

$$= \int x + \frac{2x}{x^2 - 1} + \frac{1}{x^2 - 1} dx$$

$$= \frac{x^2}{2} + \log |x^2 - 1| + \frac{1}{2} \log \left| \frac{x - 1}{x + 1} \right| + c$$

REMEMBER THIS EXERCISE AS

$$\int \frac{px + q}{\sqrt{ax^2+bx+c}}\, dx \qquad \boxed{\frac{\textbf{LINEAR}}{\sqrt{\textbf{QUADRATIC}}}}$$

Express Numerator = A $\frac{d}{dx}$(denominator) + B

$$= \int \frac{A\,\frac{d}{dx}(\text{denominator}) + B}{\sqrt{ax^2 + bx + c}}\, dx$$

$$= A\int \frac{\frac{d}{dx}(\text{denominator})}{\sqrt{ax^2 + bx + c}}\, dx \quad + \quad B\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

$$= \qquad I_1 \qquad + \qquad I_2$$

Where

$$I_1 = A\int \frac{\frac{d}{dx}(\text{denominator})}{\sqrt{ax^2 + bx + c}}\, dx \qquad \text{Using} \quad \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$$

$$= A \,.\, 2\sqrt{ax^2 + bx + c} + c_1$$

$$I_2 = B\int \frac{1}{\sqrt{ax^2 + bx + c}}\, dx$$

REFER : EX 4A

01.

I = $\int \frac{2x + 3}{\sqrt{x^2 + 4x + 1}} \, dx$

WE NEED TO EXPRESS

NUMERATOR = $A \frac{d}{dx}$ DENOMIANTOR + B

$2x + 3 = A \frac{d}{dx}(x^2 + 4x + 1) + B$

$2x + 3 = A(2x + 4) + B$

$2x + 3 = 2Ax + 4A + B$

On comparing ;

$2A = 2$ $\quad 4A + B = 3$

$A = 1$ $\quad 4.1 + B = 3$

$4 + B = 3$

$B = -1$

HENCE

$2x + 3 = 1(2x + 4) - 1$

BACK IN THE SUM

$I = \int \frac{1(2x + 4) - 1}{x^2 + 4x + 1} \, dx$

$I = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 1}} \, dx - \int \frac{1}{\sqrt{x^2 + 4x + 1}} \, dx$

$I = I_1 - I_2$

Now

$I_1 = \int \frac{2x + 3}{\sqrt{x^2 + 4x + 1}} \, dx$

$I_1 = \int \frac{f'(x)}{\sqrt{f(x)}} \, dx$

$I_1 = 2\sqrt{f(x)} + c_1$

$I_1 = 2\sqrt{x^2 + 4x + 1} + c_1$

Now

$I_2 = \int \frac{1}{\sqrt{x^2 + 4x + 1}} \, dx$

$= \int \frac{1}{\sqrt{x^2 + 4x + 4 + 1 - 4}} \, dx$

$= \int \frac{1}{\sqrt{(x + 2)^2 - 3}} \, dx$

$= \int \frac{1}{\sqrt{(x + 2)^2 - \sqrt{3}^2}} \, dx$

$= \log \left| x + \sqrt{x^2 - a^2} \right| + c_2$

$= \log \left| x + 2 + \sqrt{(x + 2)^2 - \sqrt{3}^2} \right| + c_2$

$= \log \left| x + 2 + \sqrt{x^2 + 4x + 1} \right| + c_2$

FINALLY

$I = 2\sqrt{x^2 + 4x + 1} - \log \left| x + 2 + \sqrt{x^2 + 4x + 1} \right| + c$

where $c = c_1 - c_2$

02.

I = $\int \frac{2x + 1}{\sqrt{x^2 + 4x + 3}} dx$

WE NEED TO EXPRESS

NUMERATOR = A $\frac{d}{dx}$ DENOMIANTOR + B

$2x + 1 = A \frac{d}{dx}(x^2 + 4x + 1) + B$

$2x + 1 = A(2x + 4) + B$

$2x + 1 = 2Ax + 4A + B$

On comparing ;

$2A = 2 \qquad 4A + B = 1$

$A = 1 \qquad 4.1 + B = 1$

$4 + B = 1$

$B = -3$

HENCE

$2x + 1 = 1(2x + 4) - 3$

BACK IN THE SUM

$I = \int \frac{1(2x + 4) - 3}{\sqrt{x^2 + 4x + 3}} dx$

$I = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 1}} dx - 3\int \frac{1}{\sqrt{x^2 + 4x + 1}} dx$

$I = I_1 - I_2$

Now

$I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 3}} dx$

$I_1 = \int \frac{f'(x)}{\sqrt{f(x)}} dx$

$I_1 = 2\sqrt{f(x)} + c_1$

$I_1 = 2\sqrt{x^2 + 4x + 3} + c_1$

Now

$I_2 = 3\int \frac{1}{\sqrt{x^2 + 4x + 3}} dx$

$= 3\int \frac{1}{\sqrt{x^2 + 4x + 4 + 3 - 4}} dx$

$= 3\int \frac{1}{\sqrt{(x + 2)^2 - 1}} dx$

$= 3\int \frac{1}{\sqrt{(x + 2)^2 - 1^2}} dx$

$= 3\log\left| x + \sqrt{x^2 - a^2} \right| + c_2$

$= 3\log\left| x + 2 + \sqrt{(x + 2)^2 - 1^2} \right| + c_2$

$= 3\log\left| x + 2 + \sqrt{x^2 + 4x + 3} \right| + c_2$

FINALLY

$I = 2\sqrt{x^2 + 4x + 3}$

$- 3\log\left| x + 2 + \sqrt{x^2 + 4x + 3} \right| + c$

where $c = c_1 - c_2$

03. $\int \sqrt{\dfrac{x+1}{x+2}}\, dx$

$= \int \sqrt{\dfrac{x+1}{x+2} \cdot \dfrac{x+1}{x+1}}\, dx$

$= \int \dfrac{x+1}{\sqrt{x^2+3x+2}}\, dx$

WE NEED TO EXPRESS

NUMERATOR = A $\dfrac{d}{dx}$ DENOMIANTOR + B

$x + 1 = A \dfrac{d}{dx}(x^2 + 3x + 2) + B$

$x + 1 = A(2x + 3) + B$

$x + 1 = 2Ax + 3A + B$

On comparing ;

$2A = 1 \qquad 3A + B = 1$

$A = \dfrac{1}{2} \qquad 3.\dfrac{1}{2} + B = 1$

$\dfrac{3}{2} + B = 1$

$B = 1 - \dfrac{3}{2}$

$= -\dfrac{1}{2}$

HENCE

$x + 1 = \dfrac{1}{2}(2x + 3) - \dfrac{1}{2}$

BACK IN THE SUM

$I = \int \dfrac{\dfrac{1}{2}(2x+3) - \dfrac{1}{2}}{\sqrt{x^2+3x+2}}\, dx$

$I = \dfrac{1}{2}\int \dfrac{2x+3}{\sqrt{x^2+3x+2}}\, dx - \dfrac{1}{2}\int \dfrac{1}{\sqrt{x^2+3x+2}}\, dx$

$I = I_1 - I_2$

Now

$I_1 = \dfrac{1}{2}\int \dfrac{2x+3}{\sqrt{x^2+3x+2}}\, dx$

$I_1 = \dfrac{1}{2}\int \dfrac{f'(x)}{\sqrt{f(x)}}\, dx$

$I_1 = \dfrac{1}{2} 2\sqrt{f(x)} + c_1$

$I_1 = 2\sqrt{x^2 + 4x + 3} + c_1$

Now

$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{x^2+3x+2}}\, dx$

$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{x^2 + 3x + \frac{9}{4} + 2 - \frac{9}{4}}}\, dx$

$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{\left(x + \frac{3}{2}\right)^2 + \frac{8-9}{4}}}\, dx$

$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{\left(x + \frac{3}{2}\right)^2 - \frac{1}{4}}}\, dx$

$= \dfrac{1}{2}\int \dfrac{1}{\sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$

$= \dfrac{1}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c_2$

$= \dfrac{1}{2}\log\left|x + \dfrac{3}{2} + \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c_2$

$= \dfrac{1}{2}\log\left|x + \dfrac{3}{2} + \sqrt{x^2 + 3x + 2}\right| + c_2$

FINALLY

$I = \sqrt{x^2 + 3x + 2}$

$- \dfrac{1}{2}\log\left|x + \dfrac{3}{2} + \sqrt{x^2 + 3x + 2}\right| + c$

where $c = c_1 - c_2$

04. $\int \sqrt{\dfrac{1-x}{1+x}}\ dx$

$= \int \sqrt{\dfrac{1-x}{1+x} \cdot \dfrac{1-x}{1-x}}\ dx$

$= \int \dfrac{1-x}{\sqrt{1-x^2}}\ dx$

$= \int \dfrac{1}{\sqrt{1-x^2}}\ dx - \int \dfrac{x}{\sqrt{1-x^2}}\ dx$

$= \int \dfrac{1}{\sqrt{1-x^2}}\ dx + \dfrac{1}{2} \int \dfrac{-2x}{\sqrt{1-x^2}}\ dx$

$= \sin^{-1} x + \dfrac{1}{2}\ 2\sqrt{1-x^2} + c$ ….. Using $\int \dfrac{f'(x)}{\sqrt{f(x)}}\ dx = 2\sqrt{f(x)} + c$

$= \sin^{-1} x + \sqrt{1-x^2} + c$