PAPER – I : MODEL PAPER – 01

(BASED ON MARCH 2014)

MATHEMATICS & STATISTICS

COMMERCE

SOLUTION SET

TIME : 1 HR 30 MIN **MARKS : 40**

Q1. (A) Attempt any six of the following **(12)**

01. $A = \begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$ Find matrix X such that $X = 2A + 3B$

SOLUTION

Q-1A

$$X = 2A + 3B$$

$$= 2\begin{pmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{pmatrix} + 3\begin{pmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 0 \\ 8 & -4 \\ 6 & 12 \end{pmatrix} + \begin{pmatrix} 6 & -6 \\ 12 & 6 \\ -15 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & -6 \\ 20 & 2 \\ -9 & 15 \end{pmatrix}$$

02. if the function f is continuous at $x = 2$, then find $f(2)$

$$f(x) = \frac{x^3 - 8}{x^2 - x - 2} \quad ; \quad x \neq 2$$

SOLUTION

STEP 1 $\lim\limits_{x \to 2} f(x)$

$$= \lim_{x \to 2} \frac{x^3 - 2^3}{x^2 - x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x^2 + 2x + 4)}{(x - 2)\ (x + 1)}$$

$$= \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 1} \quad , \quad x - 2 \neq 0$$

$$= \frac{4 + 4 + 4}{2 + 1} = 4$$

STEP 2 Since f is CONTINUOUS at x = 2 $\quad f(2) = \lim_{x \to 2} f(x)$

$$\therefore f(2) = 4$$

03. $x = \tan^{-1}\theta$; $y = \theta^3$; find $\dfrac{dy}{dx}$

SOLUTION

STEP 1

$$x = \tan^{-1}\theta$$

$$\frac{dx}{d\theta} = \frac{1}{1 + \theta^2}$$

STEP 2

$$y = \theta^3$$

$$\frac{dy}{d\theta} = 3\theta^2$$

STEP 3

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{3\theta^2}{\frac{1}{1 + \theta^2}}$$

$$= 3\theta 2(1 + \theta^2)$$

04. Evaluate : $\int \sin^2 3x \; dx$

SOLUTION

$$\int \sin^2 3x \; dx$$

$$1 - \cos 2x = 2\sin^2 x$$

$$\therefore \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\therefore \sin^2 3x = \frac{1 - \cos 6x}{2}$$

$$= \int \frac{1 - \cos 6x}{2} \; dx$$

$$= \frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c$$

$$= \frac{x}{2} - \frac{\sin 6x}{12} + c$$

05. Write negations of the following statements

SOLUTION

(a) policeman is honest and he is not rich

NEGATION : Using $\sim (p \wedge q) \equiv \; \sim p \vee \sim q$

policeman is not honest OR he is rich

(b) $\exists \, n \in N$, such that $n + 4 > 9$

NEGATION : $\forall \, n \in N$, $n + 4$ is not greater than 9

06. $A = \begin{bmatrix} 7 & 1 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$; Find $|AB|$

SOLUTION

$$AB = \begin{bmatrix} 7 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 7+3 & 14-1 \\ 2+15 & 4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 13 \\ 17 & -1 \end{bmatrix}$$

$$|AB| = -10 - 221 = -231$$

07. Evaluate $\int \dfrac{1}{16 - 9x^2} dx$

SOLUTION

$$\int \frac{1}{4^2 - (3x)^2} dx$$

$$= \frac{1}{3} \frac{1}{2(4)} \log \left| \frac{4+3x}{4-3x} \right| + c$$

$$= \frac{1}{24} \log \left| \frac{4+3x}{4-3x} \right| + c$$

08. $f(x) = x^2 + 1 \quad ; \quad x < 0$

$= 5\sqrt{x^2+1} + k \quad ; \quad x \geq 0$ find k if the f is continuous at x = 0

SOLUTION

STEP 1

$$\lim_{x \to 0-} f(x)$$

$$= \lim_{x \to 0} x^2 + 1$$

$$= 1$$

STEP 2

$$\lim_{x \to 0+} f(x)$$

$$= \lim_{x \to 0} 5\sqrt{x^2+1} + k$$

$$= 5(1) + k$$

$$= 5 + k$$

STEP 3

$$f(0)$$

$$= 5\sqrt{x^2+1} + k$$

$$= 5(1) + k$$

$$= 5 + k$$

STEP 4

Since f is CONTINUOUS at x = 0

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$$

$$1 = 5 + k = 5 + k \quad \therefore \; k = -4$$

Q2. (A) Attempt any TWO of the following **(06)**

Q-2A

01. If $x^y = e^x$; show that $\dfrac{dy}{dx} = \dfrac{\log x - 1}{(\log x)^2}$

SOLUTION

$x^y = e^x$

Taking log on both sides

$y \cdot \log x = x \cdot \log e$

$y \cdot \log x = x$

$y = \dfrac{x}{\log x}$

Differentiating wrt x

$$\frac{dy}{dx} = \frac{\log x \cdot \frac{d}{dx} x - x \cdot \frac{d}{dx} \log x}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x \cdot 1 - x \cdot \frac{1}{x}}{(\log x)^2}$$

$$\frac{dy}{dx} = \frac{\log x - 1}{(\log x)^2}$$ .... proved

cont.

02. if $\sin y = x.\sin(5 + y)$; prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(5 + y)}{\sin 5}$

SOLUTION

$\sin y = x.\sin(5 + y)$

$x = \dfrac{\sin y}{\sin(5 + y)}$

Differentiating wrt y

$$\frac{dx}{dy} = \frac{\sin(5 + y) \frac{d}{dy} \sin y - \sin y \frac{d}{dy} \sin(5 + y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y) \cdot \cos y - \sin y \cdot \cos(5 + y) \frac{d}{dy}(5 + y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y).\cos y - \cos(5 + y) \cdot \sin y}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin(5 + y - y)}{\sin^2(5 + y)}$$

$$\frac{dx}{dy} = \frac{\sin 5}{\sin^2(5 + y)}$$

Now $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

$\therefore \dfrac{dy}{dx} = \dfrac{\sin^2(5+ y)}{\sin 5}$

cont.

03. Discuss the extreme values of the function $f(x) = x.e^x$

SOLUTION

STEP 1 :

$$f(x) = xe^x$$

STEP 2 :

$$f'(x) = x \,.\, \frac{d}{dx} e^x + e^x \frac{d}{dx} x$$

$$= x \,.\, e^x + e^x \; 1$$

$$= e^x (x + 1)$$

$$f''(x) = e^x . \frac{d}{dx}(x + 1) + (x + 1) \frac{d}{dx} e^x$$

$$= e^x . (1) + (x + 1) . e^x$$

$$= e^x . (1 + x + 1)$$

$$= e^x . (2 + x)$$

STEP 3 :

$$f'(x) = 0$$

$$e^x (x + 1) = 0$$

$$x + 1 = 0$$

$$x = -1$$

STEP 4 :

$$f''(x)\Big|_{x = -1} = e^{-1} . (2 - 1)$$

$$= \frac{1}{e} > 0$$

f is minimum at $x = -1$

STEP 5 :

Minimum value of the f(x)

$$= f(x)\Big|_{x = -1}$$

$$= -1.e^{-1}$$

$$= \frac{-1}{e}$$

(B) Attempt any TWO of the following **(08)**

01. Discuss the continuity of the function f at x = 0

where $f(x) = \dfrac{e^x + e^{-x} - 2}{\cos 2x - \cos 6x}$; $x \neq 0$

$= \dfrac{1}{16}$; $x = 0$

Q-2B

SOLUTION

STEP 1 :

$\lim\limits_{x \to 0} f(x)$

$= \lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - 2}{\cos 2x - \cos 6x}$

$= \lim\limits_{x \to 0} \dfrac{e^x + \frac{1}{e^x} - 2}{-2 \sin \frac{2x+6x}{2} \cdot \sin \frac{2x-6x}{2}}$

$= \lim\limits_{x \to 0} \dfrac{\frac{(e^x)^2 + 1 - 2.e^x}{e^x}}{-2 \sin \frac{8x}{2} \cdot \sin \frac{-4x}{2}}$

$= \lim\limits_{x \to 0} \dfrac{\frac{(e^x - 1)^2}{e^x}}{-2 \sin 4x \cdot \sin -2x}$

$= \lim\limits_{x \to 0} \dfrac{\frac{(e^x - 1)^2}{e^x}}{2 \sin 4x \cdot \sin 2x}$

Dividing Numerator & Denominator by x^2

$x \to 0$, $x \neq 0$, $x^2 \neq 0$

$= \lim\limits_{x \to 0} \dfrac{\frac{(e^x - 1)^2}{x^2 e^x}}{\frac{2 \sin 4x \cdot \sin 2x}{x^2}}$

$= \lim\limits_{x \to 0} \dfrac{\left(\frac{e^x - 1}{x}\right)^2 \frac{1}{e^x}}{2 \frac{\sin 4x}{x} \cdot \frac{\sin 2x}{x}}$

$= \lim\limits_{x \to 0} \dfrac{\left(\frac{e^x - 1}{x}\right)^2 \frac{1}{e^x}}{2 . 4 \frac{\sin 4x}{4x} \cdot 2 \frac{\sin 2x}{2x}}$

$= \dfrac{(\log e)^2}{2.4.(1).2.(1)}$

$= \dfrac{1}{16}$

STEP 2 :

$f(0) = \dfrac{1}{16}$ .... given

STEP 3 :

$f(0) = \lim\limits_{x \to 0} f(x)$

$\therefore$ f is continuous at x = 0

02. The expenditure Ec of a person with income I is given by $E_c = 0.00002\, I^2 + 0.008\, I$ Find marginal propensity to consume (MPC) and average propensity to consume (APC) when I = 8000

SOLUTION

$$E_c = 0.00002I^2 + 0.008I$$

$$APC\Big|_{I=8000} = \frac{E_c}{I} = \frac{0.00002\, I^2 + 0.008\, I}{I}$$

$$= 0.00002\, I + 0.008$$

$$= 0.00002(8000) + 0.008$$

$$= 0.16 + 0.008$$

$$= \mathbf{0.168}$$

$$MPC\Big|_{I=8000} = \frac{dE_c}{dI} = \frac{d\, 0.00002\, I^2 + 0.008\, I}{dI}$$

$$= 0.00004\, I + 0.008$$

$$= 0.00004(8000) + 0.008$$

$$= 0.32 + 0.008$$

$$= \mathbf{0.328}$$

03. Evaluate : $\int x.\tan^{-1}x\, dx$

SOLUTION

$$= \int \tan^{-1}x.\, x\, dx$$

$$= \tan^{-1}x \int x\, dx - \int\left(\frac{d}{dx}\tan^{-1}x \int x\, dx\right)dx$$

$$= \tan^{-1}x\ .\frac{x^2}{2} - \int \frac{1}{1+x^2}.\frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}.\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx$$

$$= \frac{x^2}{2}.\tan^{-1}x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}.\, dx$$

$$= \frac{x^2}{2}.\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\, dx$$

$$= \frac{x^2}{2}.\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right) + c$$

$$= \frac{x^2}{2}.\tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + c$$

Q3. (A) Attempt any TWO of the following **(06)**

Q-3A

01. if p : Dhanashri is beautiful

q : Dhanashri is intelligent

Give the verbal statements for the following symbolic statements

a) $p \wedge \sim q$ b) $p \vee q$ c) $p \leftrightarrow q$

SOLUTION

a) $p \wedge \sim q \equiv$ Dhanashri is beautiful and she is not intelligent

b) $p \vee q \equiv$ Dhanashri is beautiful or she is intelligent

c) $p \leftrightarrow q \equiv$ Dhanashri is beautiful if and only if she is intelligent

02. Using the truth table , examine whether the statement pattern

$$(p \wedge q) \rightarrow (p \vee \sim q)$$

is a tautology , a contradiction or a contingency

SOLUTION

p	q	~ q	$p \wedge q$	$p \vee \sim q$	$(p \wedge q) \rightarrow (p \vee \sim q)$
T	T	F	T	T	T
T	F	T	F	T	T
F	T	F	F	F	T
F	F	T	F	T	T

Since all the truth values in the last column are 'T' , given statement is **'Tautology'**

03. The total cost for production of Q items is given $C = Q^3 - 600Q^2 + 1200Q$. Find the values of Q for which average cost is decreasing

SOLUTION

$C = Q^3 - 600Q^2 + 1200Q$

AVERAGE COST

$C_A = \dfrac{C}{Q}$

$= \dfrac{Q^3 - 600Q^2 + 1200Q}{Q}$

$= Q^2 - 600Q + 1200$

For average cost decreasing ,

$\dfrac{dC_A}{dQ} < 0$

$2Q - 600 < 0$

$2Q < 600$

$Q < 300$

average cost is decreasing for $Q < 300$

(B) Attempt any TWO of the following **(08)**

01. $A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Find A^{-1} by using elementary transformation

Q-3B

SOLUTION

$$\begin{aligned} |A| &= 3(5-4) - 2(5-4) + 6(2-2) \\ &= 3(1) - 2(1) + 6(0) \\ &= 3 - 2 \\ &= 1 \\ &\neq 0 \qquad \text{Hence } A^{-1} \text{ exists} \end{aligned}$$

$AA^{-1} = I$

$$\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

R_{12}

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 6 \\ 2 & 2 & 5 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 - 3R_1$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 2 & 2 & 5 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_3 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$R_2 \times (-1)$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$R_1 - R_2$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$R_1 - 2R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$I.\,A^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

02. Evaluate : $\displaystyle\int_0^2 \frac{dx}{x + \sqrt{4 - x^2}}$

SOLUTION

Put $x = 2\sin\theta$

$\dfrac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta \, d\theta$

when $x = 2$

$\sin\theta = 1$

$\theta = \pi/2$

when $x = 0$

$\sin\theta = 0$

$\theta = 0$

$$I = \int_0^{\pi/2} \frac{1}{2\sin\theta + \sqrt{4 - 4\sin^2\theta}} \, 2\cos\theta \, d\theta$$

$$I = \int_0^{\pi/2} \frac{2\cos\theta}{2\sin\theta + \sqrt{4\,(1 - \sin^2\theta)}} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{2\cos\theta}{2\sin\theta + \sqrt{4\cos^2\theta}} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{2\cos\theta}{2\sin\theta + 2\cos\theta} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} \, d\theta \;\ldots\ldots\; \textbf{(1)}$$

USING $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

WE CHANGE 'θ' **TO** '$\pi/2 - \theta$'

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - \theta)}{\cos(\pi/2 - \theta) + \sin(\pi/2 - \theta)} \, d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta} \, d\theta \;\ldots\ldots\; \textbf{(2)}$$

(1) + (2)

$$2I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta} \, d\theta$$

$$2I = \int_0^{\pi/2} 1 \, d\theta$$

$$2I = \left[\theta\right]_0^{\pi/2}$$

$$2I = \pi/2 - 0$$

$$2I = \pi/2$$

$$I = \pi/4$$

03

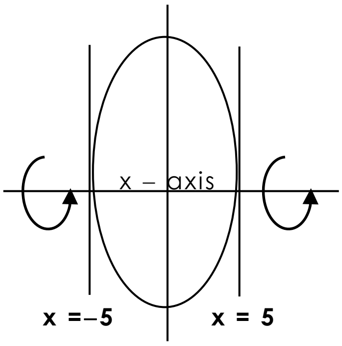
Find the volume of the solid obtained by the complete revolution of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{36} = 1 \quad \text{about } x - \text{axis}$$

STEP 1 :

$$\frac{x^2}{25} + \frac{y^2}{36} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$a^2 = 25 \; ; a = 5$

$b^2 = 36 \; , \; b = 6$

STEP 2 :

$$\frac{x^2}{25} + \frac{y^2}{36} = 1$$

$$\frac{y^2}{36} = 1 - \frac{x^2}{25}$$

$$\frac{y^2}{36} = \frac{25 - x^2}{25}$$

$$y^2 = \frac{36}{25}(25 - x^2)$$

STEP 3 :

$$V = \pi \int_{-5}^{5} y^2.dx$$

$$= \pi \int_{-5}^{5} \frac{36}{25}(25 - x^2).dx$$

$$= \frac{36\pi}{25} \int_{-5}^{5} (25 - x^2).dx$$

$$= \frac{36\pi}{25} \left[25x - \frac{x^3}{3}\right]_{-5}^{5}$$

$$= \frac{36\pi}{25} \left\{\left[125 - \frac{125}{3}\right] - \left[-125 + \frac{125}{3}\right]\right\}$$

$$= \frac{36\pi}{25} \left\{\left[\frac{375 - 125}{3}\right] - \left[\frac{-375 + 125}{3}\right]\right\}$$

$$= \frac{36\pi}{25} \left\{\left[\frac{250}{3}\right] - \left[\frac{-250}{3}\right]\right\}$$

$$= \frac{36\pi}{25} \left[\frac{500}{3}\right]$$

$= 240\,\pi$ cubic units

DO NOT STOP

GET READY FOR NEXT