PAPER - I : MODEL PAPER - 04

(BASED ON MARCH 2016**)**

MATHEMATICS & STATISTICS

COMMERCE

TIME : 1 HR 30 MIN **MARKS : 40**

Q1. (A) Attempt any six of the following **(12)**

01. Find $\frac{dy}{dx}$ if $y = (\sin x)^x$

SOLUTION

$$\log y = x.\log(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}x$$

$$\frac{dy}{dx} = y\left(x\frac{1}{\sin x}\frac{d}{dx}\sin x + \log(\sin x)\right)$$

$$\frac{dy}{dx} = y\left(x\frac{\cos x}{\sin x} + \log(\sin x)\right)$$

$$\frac{dy}{dx} = (\sin x)^x\left(x.\cot x + \log(\sin x)\right)$$

02. if $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ Show that : $A^2 - 2A$ is a scalar matrix

SOLUTION $A^2 - 2A$

$$= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9 & 3+3 \\ 3+3 & 9+1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$ is a scalar matrix ………… PROVED

03. Write negations of the following statements

1. $\forall\, n \in N,\ n + 7 > 8$

NEGATION : $\exists\, n \in N$, such that $n + 7$ is not greater than 8

2. If it snows then Gajashri does not drive the car

Using : $\sim(P \rightarrow Q) \equiv P \wedge \sim Q$

Negation : It snows and Gajashri does drive the car

04. For manufacturing x units , price of each unit is $p = 10800 - 4x^2$ .Find the values of x for which revenue is increasing

SOLUTION : $R = px$

$= 10800x - 4x^3$

For Revenue increasing $\dfrac{dR}{dx} > 0$

$10800 - 12x^2 > 0$

$10800 > 12x^2$

$x^2 < 900$

$x < 30$

05. Evaluate : $\displaystyle\int \frac{\sec x.\tan x}{\sec^2 x + 4}\, dx$

SOLUTION : $\sec x = t$

$\sec x \,.\, \tan x \,.\, dx = dt$

$= \displaystyle\int \frac{1}{t^2 + 2^2}\, dt$

$= \dfrac{1}{2} \tan^{-1}\left(\dfrac{t}{2}\right) + c$

$= \dfrac{1}{2} \tan^{-1}\left(\dfrac{\sec x}{2}\right) + c$

06. Find $\dfrac{dy}{dx}$ if $y = \sin^{-1}(\cos 3x)$

SOLUTION : $y = \sin^{-1}(\cos 3x)$

$y = \sin^{-1}\sin(\pi/2 - 3x)$

$y = \pi/2 - 3x$

Differentiating wrt x

$\dfrac{dy}{dx} = -3$

07. $f(x) = \dfrac{\sqrt{x+3} - 2}{x^3 - 1}$; $x \neq 1$

$= 5$; $x = 1$ Discuss continuity at $x = 1$

SOLUTION : STEP1 :

$$\lim_{x \to 1} f(x)$$

$$= \lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x^3 - 1} \cdot \frac{\sqrt{x+3} + 2}{\sqrt{x+3} + 2}$$

$$= \lim_{x \to 1} \frac{x + 3 - 4}{(x-1)(x^2 + x + 1)} \cdot \frac{1}{\sqrt{x+3} + 2}$$

$$= \lim_{x \to 1} \frac{x - 1}{(x-1)(x^2 + x + 1)} \cdot \frac{1}{\sqrt{x+3} + 2}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + x + 1} \cdot \frac{1}{\sqrt{x+3} + 2}$$

$$= \frac{1}{1 + 1 + 1} \cdot \frac{1}{2 + 2}$$

$$= \frac{1}{12}$$

STEP2 : $f(1) = 5$ Given

STEP3 : $f(1) \neq \lim_{x \to 1} f(x)$; f is DISCONTINUOUS at $x = 1$

STEP4 : f can be made continuous by redefining it as

$f(x) = \dfrac{\sqrt{x+3} - 2}{x^3 - 1}$; $x \neq 1$

$= {}^1/_{12}$; $x = 1$

08. State which of the following sentences are statements . In case of statement , write down the truth value

a) Every quadratic equation has only real roots

It is a logical statement . Truth value : F

b) $\sqrt{-4}$ is a rational number

It is a logical statement . Truth value : F

Q2. (A) Attempt any TWO of the following **(06)**

01. Solve the following equations by the inversion method

$$2x + 3y = -5 \quad \text{and} \quad 3x + y = 3$$

SOLUTION :

STEP 1 :

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$AX = B$

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

STEP 2 :

$$A = \begin{matrix} 2 & 3 \\ 3 & 1 \end{matrix}$$

COFACTORS

$A_{11} = (-1)^{1+!}(1) = 1$

$A_{12} = (-1)^{1+2}(3) = -3$

$A_{21} = (-1)^{2+1}(3) = -3$

$A_{22} = (-1)^{2+2}(2) = 2$

COFACTOR MATRIX

$$\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{-7}\begin{pmatrix} 1 & -3 \\ -3 & 2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$$

STEP 3 :

$$X = A^{-1}B$$

$$= \frac{1}{7}\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 5 + 9 \\ -15 - 6 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 14 \\ -21 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

BY EQUALITY OF MATRICES

$x = 2$ & $y = -3$

02 .

$$\left\{ 2\begin{bmatrix} 5 & 0 & -1 \\ 1 & 2 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \right\} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find x and y

SOLUTION :

$$\left\{ 2\begin{bmatrix} 5 & 0 & -1 \\ 1 & 2 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \right\} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left\{ \begin{bmatrix} 10 & 0 & -2 \\ 2 & 4 & -6 \end{bmatrix} - \begin{bmatrix} 6 & -3 & 3 \\ -12 & 6 & 9 \end{bmatrix} \right\} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 4 & 3 & -5 \\ 14 & -2 & -15 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} 4 + 3 - 0 \\ 14 - 2 - 0 \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 7 \\ 12 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

By Equality of Matrices ; $x = 7$, $y = 12$

03. Evaluate : $\int \cot^{-1}x \; dx$

SOLUTION

$$\cot^{-1}x \int 1dx - \int \frac{d}{dx}\cot^{-1}x \int 1dx \; dx$$

$$\cot^{-1}x \, . \, x - \int \frac{-1}{1 + x^2} \; x \; dx$$

$$x. \cot^{-1}x + \int \frac{x}{1 + x^2} \, dx$$

$$x. \cot^{-1}x + \frac{1}{2}\int \frac{2x}{1 + x^2} \, dx$$

$$x. \cot^{-1}x + \frac{1}{2} \log |1 + x^2| + c$$

(B)Attempt any TWO of the following **(08)**

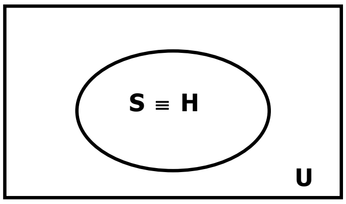
01. a) Express the truth of each of the following statements using Venn Diagram

1. All Sundays are holidays and holidays are Sundays

S ≡ set of all Sunday's

H ≡ set of all holiday's

U ≡ set of all days

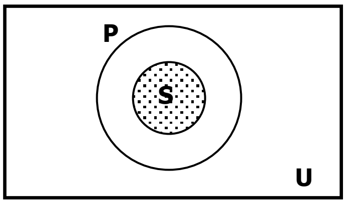


2. if a quadrilateral is a square then it is a parallelogram

S ≡ set of all squares

P ≡ set of all parallelograms

U ≡ set of all quadrilaterals



b) Write converse and inverse of the following statement

'If function is differentiable then it is continuous'

SOLUTION

$P \rightarrow Q$ ≡ If function is differentiable then it is continuous

Converse : $Q \rightarrow P$

If function is continuous then it is differentiable

Contrapositive : $\sim Q \rightarrow \sim P$

If function is not continuous then it is not differentiable

Inverse : $\sim P \rightarrow \sim Q$

If function is not differentiable then it is not continuous

02. Find the volume of the solid obtained by the complete revolution of the ellipse

$$\frac{x^2}{36} + \frac{y^2}{25} = 1 \quad \text{about } y - \text{axis}$$

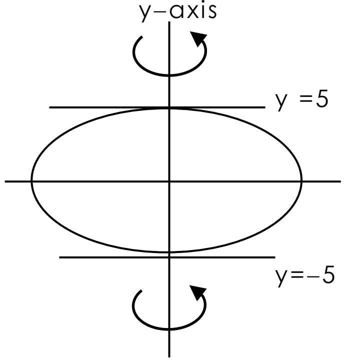
STEP 1 :

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$a^2 = 36$; $a = 6$

$b^2 = 25$, $b = 5$



STEP 2 :

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

$$\frac{x^2}{36} = 1 - \frac{y^2}{25}$$

$$\frac{x^2}{36} = \frac{25 - y^2}{25}$$

$$x^2 = \frac{36}{25}(25 - y^2)$$

STEP 3 :

$$V = \pi \int_{-5}^{5} x^2 . dy$$

About y – axis

$$= \pi \int_{-5}^{5} \frac{36}{25}(25 - y^2).dy$$

$$= \frac{36\pi}{25} \int_{-5}^{5} (25 - y^2).dy$$

$$= \frac{36\pi}{25} \left(25y - \frac{y^3}{3}\right)_{-5}^{5}$$

$$= \frac{36\pi}{25} \left\{\left(125 - \frac{125}{3}\right) - \left(-125 + \frac{125}{3}\right)\right\}$$

$$= \frac{36\pi}{25} \left\{\left(\frac{375 - 125}{3}\right) - \left(\frac{-375 + 125}{3}\right)\right\}$$

$$= \frac{36\pi}{25} \left\{\left(\frac{250}{3}\right) - \left(\frac{-250}{3}\right)\right\}$$

$$= \frac{36\pi}{25} \left(\frac{500}{3}\right)$$

$= 240\,\pi$ cubic units

Q3. (A) Attempt any TWO of the following **(06)**

01. find a & b if f(x) is continuous at x = 0

$$f(x) = \frac{e^{2x}-1}{ax} \quad ; \quad x < 0 , a \neq 0$$
$$= 1 \quad ; \quad x = 0$$
$$= \frac{\log(1+7x)}{bx} \quad ; \quad x > 0 , b \neq 0$$

STEP 1

$$\lim_{x\to 0-} f(x)$$

$$= \lim_{x\to 0} \frac{e^{2x}-1}{ax}$$

$$= \lim_{x\to 0} \frac{1}{a}\,\frac{e^{2x}-1}{x}$$

$$= \lim_{x\to 0} \frac{2}{a}\,\frac{e^{2x}-1}{2x}$$

$$= \frac{2.\log e}{a} = 2/a$$

STEP 2

$$\lim_{x\to 0+} f(x)$$

$$= \lim_{x\to 0} \frac{\log(1+7x)}{bx}$$

$$= \lim_{x\to 0} \frac{1}{b}\,\frac{\log(1+7x)}{x}$$

$$= \lim_{x\to 0} \frac{7}{b}\,\frac{\log(1+7x)}{7x}$$

$$= \frac{7}{b}(1) = \frac{7}{b}$$

STEP 3

f(0) = 1 ……… given

STEP 4

Since *f* is continuous at x = 0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$\frac{2}{a} = \frac{7}{b} = 1$$

$\therefore$ a = 2 & b = 7

02.
$$f(x) = \frac{\sin^2 x}{1-\cos^3 x} \quad ; \quad x \neq 0$$
$$= 3/2 \quad ; \quad x = 0$$

Discuss continuity at x = 0

STEP 1

$$\lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{1-\cos^3 x}$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{(1-\cos x)(1+\cos x+\cos^2 x)}$$

$$= \lim_{x\to 0} \frac{\cancel{(1-\cos x)}(1+\cos x)}{\cancel{(1-\cos x)}(1+\cos x+\cos^2 x)}$$

as $x\to 0$, $1-\cos x \neq 0$

$$= \lim_{x\to 0} \frac{1+\cos x}{1+\cos x+\cos^2 x}$$

$$= \frac{1+\cos 0}{1+\cos 0+\cos^2 0}$$

$$= \frac{1+1}{1+1+1} = \frac{2}{3}$$

STEP 2 :

f(0) = 1 ……… given

STEP 3 :

$$f(0) \neq \lim_{x\to 0} f(x)$$

$\therefore$ *f* is discontinuous at x = 0

STEP 4 :

Removable Discontinuity

f can be made continuous at x = 0 by redefining it as

$$f(x) = \frac{\sin^2 x}{1-\cos^3 x} \quad ; \quad x \neq 0$$
$$= 2/3 \quad ; \quad x = 0$$

03. if $f'(x) = 4x^3 - 3x^2 + 2x + k$ and $f(0) = 1$, $f(1) = 4$. Find $f(x)$

$$f(x) = \int f'(x)dx$$

$$= \int (4x^3 - 3x^2 + 2x + k)\, dx$$

$$= \frac{4x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2} + kx + c$$

$$f(x) = x^4 - x^3 + x^2 + kx + c$$

$$f(0) = 1$$

$$0 - 0 + 0 + 0 + c = 1$$

$$c = 1$$

$$f(1) = 4$$

$$1 - 1 + 1 + k + c = 4$$

$$k + c = 3$$

$$k + 1 = 3$$

$$k = 2$$

Hence

$$f(x) = x^4 - x^3 + x^2 + 2x + 1$$

(B) Attempt any TWO of the following **(08)**

01. Discuss the extreme values of $f(x) = x.\log x$

SOLUTION

STEP 1 :

$$f(x) = x.\log x$$

STEP 2 :

$$f'(x) = x . \frac{d}{dx} \log x + \log x \frac{d}{dx} x$$

$$= x . \frac{1}{x} + \log x$$

$$= 1 + \log x$$

$$f''(x) = \frac{1}{x}$$

STEP 3 :

$$f'(x) = 0$$

$$1 + \log x = 0$$

$$\log x = -1$$

$$x = e^{-1}$$

STEP 4 :

$$f''(x) = \frac{1}{e^{-1}}$$

$$= e > 0$$

f is minimum at $x = e^{-1}$

STEP 5 :

Minimum value of the $f(x)$

$$= f(x)\Big|_{x = -1}$$

$$= e^{-1}.\log e^{-1}$$

$$= \frac{1}{e} . -1 \log e$$

$$= \frac{-1}{e}$$

02. in a firm the cost function for output x is given as $C = \frac{x^3}{3} - 20x^2 + 70x$

Find the output for which Marginal cost is minimum

SOLUTION

STEP 1 : MARGINAL COST

$$C_M = \frac{dC}{dx} = x^2 - 40x + 70$$

STEP 2 :

$$\frac{dC_M}{dx} = 2x - 40$$

$$\frac{d^2C_M}{dx^2} = 2$$

STEP 3 :

$$\frac{dC_M}{dx} = 0$$

$$2x - 40 = 0 \quad \therefore \quad x = 20$$

STEP 4 :

$$\left.\frac{d^2C_M}{dx^2}\right|_{x = 20} = 2 > 0$$

Marginal cost is minimum at x = 20

03. $\log \left(\frac{x^4 - y^4}{x^4 + y^4}\right) = k$

Show that : $\frac{dy}{dx} = \frac{y}{x}$

SOLUTION

$$\log \left(\frac{x^4 - y^4}{x^4 + y^4}\right) = k$$

$$\left(\frac{x^4 - y^4}{x^4 + y^4}\right) = e^k$$

$$\left(\frac{x^4 - y^4}{x^4 + y^4}\right) = m \text{ (say)}$$

$$x^4 - y^4 = mx^4 + my^4$$

$$x^4 - mx^4 = my^4 + y^4$$

$$x^4 (1 - m) = y^4 (m + 1)$$

$$y^4 = x^4 \frac{1 - m}{1 + m}$$

Differentiating wrt x

$$4y3 \frac{dy}{dx} = 4x^3 \frac{1 - m}{1 + m}$$

$$\frac{dy}{dx} = \frac{x^3}{y^3} \frac{1 - m}{1 + m}$$

$$\frac{dy}{dx} = \frac{x^3}{y^3} \frac{y4}{x4}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

………. PROVED

DO NOT STOP

GET READY FOR NEXT